INTERPOLATED FREE GROUP FACTORS

Ken Dykema

University of California,
Berkeley, California, USA 94720,
(e–mail dykema@math.berkeley.edu)

March 1992

Abstract. The interpolated free group factors $L(\mathbf{F}_r)$ for $1 < r \leq \infty$, (also defined by F. Rădulescu) are given another (but equivalent) definition as well as proofs of their properties with respect to compression by projections and free products. In order to prove the addition formula for free products, algebraic techniques are developed which allow us to show $R*R \cong L(\mathbf{F}_2)$ where R is the hyperfinite II_1–factor.

Introduction.

The free group factors $L(\mathbf{F}_n)$ for $n = 2, 3, \dots, \infty$ (introduced in [4]) have recently been extensively studied [11,2,5,6,7] using Voiculescu's theory of freeness in noncommutative probability spaces (see [8,9,10,11,12,13], especially the latter for an overview). One hopes to eventually be able to solve the old isomorphism question, first raised by R.V. Kadison in the 1960's, of whether $L(\mathbf{F}_n) \cong L(\mathbf{F}_m)$ for $n \neq m$. In [7], F. Rădulescu introduced II_1–factors $L(\mathbf{F}_r)$ for $1 < r \leq \infty$, equaling the free group factor $L(\mathbf{F}_n)$ when $r = n \in \mathbf{N}\backslash\{0,1\}$ and satisfying

$$L(\mathbf{F}_r) * L(\mathbf{F}_{r'}) = L(\mathbf{F}_{r+r'}),\ (1 < r, r' \leq \infty) \tag{1}$$

and

$$L(\mathbf{F}_r)_\gamma = L(\mathbf{F}(1 + \frac{r-1}{\gamma^2})),\ (1 < r \leq \infty,\ 0 < \gamma < \infty). \tag{2}$$

Where for a II_1–factor $\mathcal{M}$, $\mathcal{M}_\gamma$ means the algebra [4] defined as follows: for $0 < \gamma \leq 1$, $\mathcal{M}_\gamma = p\mathcal{M}p$, where $p \in \mathcal{M}$ is a self–adjoint projection of trace γ; for $\gamma = n = 2, 3, \dots$ one has $\mathcal{M}_\gamma = \mathcal{M} \otimes M_n(\mathbf{C})$; for $0 < \gamma_1, \gamma_2 < \infty$ one has

$$\mathcal{M}_{\gamma_1\gamma_2} = (\mathcal{M}_{\gamma_1})_{\gamma_2}.$$

We had independently found the interpolated free group factors $L(\mathbf{F}_r)$ $(1 < r \leq \infty)$ and the formulas (1) and (2), defining them differently and using different techniques. In this paper we give our definition and proofs. This picture of $L(\mathbf{F}_r)$ is sometimes more convenient, *e.g.* §4 of [3]. It is a natural extension of the result [2] that

$$\text{Ł}(\mathbf{Z}) * R \cong \text{Ł}(\mathbf{F}_2), \tag{3}$$

where R is the hyperfinite II_1–factor. We also introduce some elementary algebraic techniques for freeness which have further application in [3]. One consequence of them that we prove here is that $R * R \cong \text{Ł}(\mathbf{F}_2)$.

This paper has four sections. In §1 we state a random matrix result (from [2], [7]) and some consequences; in §2 we define the interpolated free group factors and prove the formula (2); in §3 we develop the algebraic techniques;

Studies and research supported by the Fannie and John Hertz Foundation.
This work will form part of the author's Ph.D. thesis at the University of California, Berkeley.

in §4 we prove the addition formula (1) and also make an observation from (1) and (2) (also observed in [7]) that, as regards the isomorphism question, we must have one of two extremes. Our original proof of the addition formula (1) was a fairly messy application of the algebraic techniques developed in §3. The proof of Theorem 4.1 that appears here, while still using the algebraic techniques in an essential way, benefits significantly from ideas found in the proof of F. Rădulescu [7].

§1. The matrix model.

Voiculescu, as well as developing the whole notion of freeness in noncommutative probability spaces, had the fundamental idea of using Gaussian random matrices to model freeness, which he developed in [12]. In [2], we extended this matrix model to the non–Gaussian case and also to be able to handle semicircular families together with a free finite dimensional algebra. As Rădulescu observed in [7], the matrix model necessary to be able to handle the free finite dimensional algebra can be easily proved in the Gaussian case directly using Voiculescu's methods (*cf.* the appendix of [2]). In any case, we shall use this matrix model in this paper, and quote it here, as well as some results of it. Our notation for random matrices will be as in [2]. A trivial reformulation of Theorem 2.1 of [2] gives

Theorem 1.1. *Let $Y(s,n) \in M_n(L)$ for $s \in S$ be self–adjoint independently distributed $n \times n$ random matrices as in Theorem 2.1 of* [2]. *For $c = \begin{pmatrix} c_{11} & \cdots & c_{1N} \\ \vdots & \ddots & \vdots \\ c_{N1} & \cdots & c_{NN} \end{pmatrix} \in M_N(\mathbf{C})$ and for n a multiple of N let $c(n) = \begin{pmatrix} c_{11}I_{\frac{n}{N}} & \cdots & c_{1N}I_{\frac{n}{N}} \\ \vdots & \ddots & \vdots \\ c_{N1}I_{\frac{n}{N}} & \cdots & c_{NN}I_{\frac{n}{N}} \end{pmatrix}$ be a constant matrix in $M_n(L)$. Then*

$$\big\{(\{Y(s,n)\})_{s\in S}, \{c(n) \mid c \in M_N(\mathbf{C})\}\big\}$$

is an asymptotically free family as $n \to \infty$, and each $Y(s,n)$ has for limit distribution a semicircle law.

An immediate result of the above is (3.2 of [2])

Theorem 1.2. *In a noncommutative probability space $(\mathcal{M}, \phi)$ with ϕ a trace, let $\nu_1 = \{X^s \mid s \in S\}$ be a semicircular family and let $\nu_2 = \{e_{ij} \mid 1 \le i,j \le n\}$ be a system of matrix units such that $\{\nu_1, \nu_2\}$ is free. Then in $(e_{11}\mathcal{M}e_{11}, n\phi|_{e_{11}\mathcal{M}e_{11}})$, $\omega_1 = \{e_{1i}X^s e_{i1} \mid 1 \le i \le n,\ s \in S\}$ is a semicircular family and $\omega_2 = \{e_{1i}X^s e_{j1} \mid 1 \le i < j \le n,\ s \in S\}$ is a circular family such that $\{\omega_1, \omega_2\}$ is free.*

The following is analogous to Theorem 2.4 of [11].

Theorem 1.3. *In a noncommutative probability space $(\mathcal{M}, \phi)$ with ϕ a trace, let $\nu = \{X^s \mid s \in S\}$ be a semicircular family and let R be a copy of the hyperfinite II_1–factor such that $\{\nu, R\}$ is free. Let $p \in R$ be a nonzero self–adjoint projection. Then in $(p\mathcal{M}p, \phi(p)^{-1}\phi|_{p\mathcal{M}p})$, $\omega = \{pX^s p \mid s \in S\}$ is a semicircular family and $\{pRp, \omega\}$ is free. (Note from* [4] *that pRp is also a copy of the hyperfinite II_1–factor.)*

Proof. Suppose first that $\phi(p) = m/2^k$, a dyadic rational number. Since for $U \in R$ a unitary, $\{R, U\nu U^*\}$ is free, we may let p be any projection in R of the given trace. Writing $R = M_{2^k} \otimes M_2 \otimes M_2 \otimes \cdots$, we use Theorem 1.1 in order to model ν as the limit of self-adjoint independently distributed random matrices of size $n = 2^k, 2^{k+1}, 2^{k+2}, \ldots$, and model a dense subalgebra of R (equal to the tensor product of matrix algebras) by constant random matrices. Choosing p to correspond to a diagonal element of M_{2^k}, we may apply Theorem 1.1 again to see that ω is a semicircular family, $pRp \cong M_m \otimes M_2 \otimes M_2 \otimes \cdots$, and $\{pRp, \omega\}$ is free.

Now for general p, let $(p_l)_{l=1}^{\infty}$ be a decreasing sequence of projections in R which converge to p and such that each $\phi(p_l)$ is a dyadic rational number. Then

$$\big\{p_l R p_l = \{p_l y p_l \mid y \in R\}, \{p_l X^s p_l \mid s \in S\}\big\}$$

has limit distribution equal to $\{pRp, \omega\}$ as $l \to \infty$. For each l we have freeness and semicircularity, hence also in the limit. $\square$

In addition, modeling R and a semicircular family as in the above proof, we can easily prove

Theorem 1.4. *In a noncommutative probability space $(\mathcal{M}, \phi)$ with ϕ a trace, let $\nu = \{X^s \mid s \in S\}$ be a semicircular family, and R a hyperfinite II_1–factor containing a system of matrix units $\{e_{ij} \mid 1 \le i, j \le n\}$, such that $\{\nu, R\}$ is free. Then in $(e_{11}\mathcal{M}e_{11}, n\phi|_{e_{11}\mathcal{M}e_{11}})$, $\omega_1 = \{e_{1i}X^s e_{i1} \mid 1 \le i \le n,\ s \in S\}$ is a semicircular family and $\omega_2 = \{e_{1i}X^s e_{j1} \mid 1 \le i < j \le n,\ s \in S\}$ is a circular family such that $\{\omega_1, \omega_2, e_{11}Re_{11}\}$ is $*$–free.*

§2. Definition and compressions of $L(\mathbf{F}_r)$.

Definition 2.1. In a W*–probability space $(\mathcal{M}, \tau)$, where τ is a faithful trace, let R be a copy of the hyperfinite II$_1$–factor and $\omega = \{X^t \mid t \in T\}$ be a semicircular family such that R and ω are free. Then $L(\mathbf{F}_r)$ for $1 < r \le \infty$ will denote any factor isomorphic to $(R \cup \{p_t X^t p_t \mid t \in T\})''$, where $p_t \in R$ are self–adjoint projections and $r = 1 + \sum_{t \in T} \tau(p_t)^2$.

Proposition 2.2. *$L(\mathbf{F}_r)$ is well–defined,* i.e. *if $\mathcal{A} = (R \cup \{p_t X^t p_t \mid t \in T\})''$ and $\mathcal{B} = (R \cup \{q_t X^t q_t \mid t \in T\})''$, where $1 + \sum \tau(p_t)^2 = r = 1 + \sum \tau(q_t)^2$, then $\mathcal{A} \cong \mathcal{B}$.*

Proof. We show that $\mathcal{A}$ (and thus also $\mathcal{B}$) is isomorphic to an algebra of a certain "standard form." Let $(f_k)_{k=1}^{\infty}$ be an orthogonal family of projections in R such that $\tau(f_k) = 2^{-k}$, and let $f_0 = 1$. We show that $\mathcal{A}$ is isomorphic to $\mathcal{C} = (R \cup \{f_{k_s} X^s f_{k_s} \mid s \in S\})''$, where $S \subseteq T$, each $k_s \in \mathbf{N} = \{0, 1, 2, \dots\}$, $1 + \sum_{s \in S} \tau(f_{k_s})^2 = r$ and for each $k \ge 0$, letting $S(k) = \{s \in S \mid k_s > k\}$ we have that $\sum_{s \in S(k)} 2^{-k_s} < 2^{-k}$. (Note that these conditions imply a unique choice of $|\{s \mid k_s = k\}|$ for all k.) This will prove the proposition.

Proving $\mathcal{A} \cong \mathcal{C}$ is an exercise in cutting and pasting. Note that if U_t are unitaries in R, $(t \in T)$, then $\{R, (\{U_t X^t U_t^*\})_{t \in T}\}$ is free in $(\mathcal{M}, \tau)$. Moreover, each projection $p \in R$ is conjugate by a unitary in R to a projection that is a (possibly infinite) sum of projections in $\{f_k \mid k \ge 1\}$. Hence letting $T' = \{t \in T \mid p_t \ne 0\}$, we may assume without loss of generality that each p_t for $t \in T'$ is equal to such a sum, and we write $p_t = \sum_{k \in K_t} f_k$, for $K_t \subseteq \mathbf{N} \backslash \{0\}$ whenever $t \in T'$ and $p_t \ne 1$, and we set $K_t = \{0\}$ if $p_t = 1$. Then

$$\mathcal{A} = \left(R \cup \{f_k X^t f_{k'} \mid k, k' \in K_t,\ k' \le k,\ t \in T'\}\right)''.$$

Now we may appeal to the matrix model (§1) to see that (enlarging T if necessary),

$$\mathcal{A} = \left(R \cup \{f_k X^{\alpha(k,k',t)} f_{k'} \mid k, k' \in K_t,\ k' \le k,\ t \in T'\}\right)'',$$

where α is a 1–1 map from $\{k, k' \in K_t,\ k' \le k,\ t \in T'\}$ onto a subset T'' of T. (The truth of the above assertion is most easily demonstrated when T' and each K_t are finite; the general case then follows by taking inductive limits.)

Consider for a moment $f_k X^t f_{k'}$ for $k' < k$, $t \in T$. Note that $f_{k'}$ is the sum of $2^{k-k'}$ orthogonal projections, each of which is equivalent in R to f_k. Using the matrix model shows that

$$(R \cup \{f_k X^t f_{k'}\})'' \cong \left(R \cup \{f_k X^{t_j} f_k \mid 1 \le j \le 2^{k-k'}\} \cup \{f_k X^{t'_j} f_k \mid 1 \le j \le 2^{k-k'}\}\right)'', \tag{4}$$

where $t_1, \dots, t_{2^{k-k'}}, t'_1, \dots, t'_{2^{k-k'}}$ are distinct elements of T, and the isomorphism in (4) maps R identically onto itself. Using inductive limits, one obtains

$$\mathcal{A} \cong \widetilde{\mathcal{C}} = \left(R \cup \{f_{k_s} X^s f_{k_s} \mid s \in S'\}\right)'', \tag{5}$$

for S' some subset of T, $k_s \in \mathbf{N}$ for each $s \in S'$. Moreover, checking the arithmetic of the above moves shows that $1 + \sum_{s \in S'} \tau(f_{k_s})^2 = r$.

Now for the pasting. Note that by the matrix model,

$$(R \cup \{f_k X^{t_i} f_k \mid 1 \le i \le 4\})'' \cong (R \cup \{f_{k-1} X^t f_{k-1}\})'' \tag{6}$$

by an isomorphism mapping R identically to itself, whenever $k \ge 1$, $t_1, \dots, t_4$ are distinct elements of T and $t \in T$. Suppose $r < \infty$. Now using (6) and induction, we can show that $\widetilde{\mathcal{C}} \cong (R \cup \{f_{k_s} X^s f_{k_s} \mid s \in S''\})''$ for some $S'' \subseteq T$,

$k_s \in \mathbf{N}$ for each $s \in S''$, and where $S''_k = \{s \in S'' \mid k_s = k\}$ has cardinality 0, 1, 2 or 3 whenever $k \in \mathbf{N}\backslash\{0\}$. (This is a complicated induction argument, which terminates because $r < \infty$.) We need now be concerned only with the case $|S''_k| = 3 \ \forall k \geq k'$, some fixed $k' > 0$. In this case, to show that $\widetilde{\mathcal{C}}$ is isomorphic to an algebra $\mathcal{C}$ that is in standard form, it will suffice to show

$$\big(R \cup \{f_{k_s} X^s f_{k_s} \mid s \in \bigcup_{k \geq k'} S''_k\}\big)'' \cong (R \cup \{f_{k'-1} X^t f_{k'-1}\})'', \tag{7}$$

where $t \in T$ and where the isomorphism maps R identically to itself. With the matrix model, we easily see that for $N \geq k'$,

$$\big(R \cup \{f_{k_s} X^s f_{k_s} \mid s \in \bigcup_{k' \leq k \leq N} S''_k\}\big)'' \cong (R \cup \{f_{k'-1} X^t f_{k'-1} - f_N X^t f_N\})'',$$

and using inductive limits gives (7).

If $r = \infty$, then considering S' from (5) and letting $S'_k = \{s \in S' \mid k_s = k\}$, we have that $\sum_{k=0}^{\infty} |S'_k| 4^{-k} = \infty$. Now by repeated application of (6), we can transform the situation (by isomorphisms mapping R identically to itself) so that first some $|S'_k| = \infty$, then all $|S'_k| = \infty$, then $|S'_0| = \infty$ and $|S'_k| = 0$ for all $k \geq 1$. Thus $\widetilde{\mathcal{C}} = L(\mathbf{F}_\infty)$ by (3). □

Remark 2.3. Formula (2), together with the fact that $L(\mathbf{F}_r)$ for $r \in \mathbf{N}$ is the free group factor on r generators, shows that Definition 2.1 is equivalent to Rădulescu's definitions 4.1 and 5.3 of [7]. However, for $r \geq 2$, (*i.e.* Rădulescu's 4.1), this equivalence can be seen directly using the "standard form" of $L(\mathbf{F}_r)$ as defined in Proposition 1.3, and by noting that the isomorphism

$$R * L(\mathbf{Z}) \xrightarrow{\sim} L(\mathbf{Z}) * L(\mathbf{Z}) \tag{8}$$

in [2] sends the set of projections $\{f_k \mid k \geq 1\} \subset R$ into one of the copies of $L(\mathbf{Z})$ on the right hand side of (8).

The formula in the following theorem for the compression of an interpolated free group factor $L(\mathbf{F}_r)$ by a projection of trace γ was first proved by Voiculescu [11] for the cases $r = 2, 3, \ldots,\ \gamma = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and $r = \infty$, $\gamma \in \mathbf{Q}_+$. It was then extended by F. Rădulescu in [5] for $r = \infty$ and $\gamma \in \mathbf{R}_+$, and in [6] for $r = 2, 3, \ldots$ and $\gamma = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots$. Of course, Rădulescu also proved this theorem in the generality stated here in [7].

Theorem 2.4.

$$L(\mathbf{F}_r)_\gamma = L(\mathbf{F}(1 + \frac{r-1}{\gamma^2})) \tag{9}$$

for $1 < r \leq \infty$ *and* $0 < \gamma < \infty$.

Proof. It suffices to show the case $0 < \gamma < 1$. Let $L(\mathbf{F}_r) = \mathcal{A} = (R \cup \{p_t X^t p_t \mid t \in T\})''$ be as in Definition 2.1, so $1 + \sum_{t \in T} \tau(p_t)^2 = r$. Let $p \in R$ be a projection having trace γ. Without loss of generality, we may assume that each $p_t \leq p$. Then $p\mathcal{A}p = (pRp \cup \{p_t X^t p_t \mid t \in T\})''$, which by Theorem 1.3 is an interpolated free group factor. Counting gives the formula (9). □

§3. Algebraic techniques.

A crucial ingredient of our proof of the addition formula for free products (1) will be showing that $R*R \cong R*L(\mathbf{Z})$, with the isomorphism being the identity map on the first copy of R. In order to show this, we will introduce some elementary techniques (Definition 3.4, proof of Theorem 3.5) that are algebraic in nature. These techniques have extensive further applications to free products, as will be seen in [3].

Remark 3.1. In this section, all von Neumann algebras will be finite and have fixed normalized faithful traces associated to them, and all isomorphisms and inclusions of von Neumann algebras will be assumed to be trace preserving. Von Neumann algebras that we obtain from others by certain operations will have associated traces given by the following conventions:

(1) group von Neumann algebras $L(G)$ for G a discrete group will have their canonical traces (equal to the vector–state for the vector $\delta_e \in l^2(G)$);

(2) factors, such as matrix algebras $M_n = M_n(\mathbf{C})$ or the hyperfinite II_1–factor R, will have (of course) their unique normalized traces;
(3) a tensor product $A \otimes B$ of algebras will have the tensor product trace $\tau_A \otimes \tau_B$ of the given traces on A and B;
(4) a free product $A * B$ of algebras will have the free product trace $\tau_A * \tau_B$ of the given traces on A and B;
(5) if $\mathcal{M}$ is a von Neumann algebra with faithful trace τ, and p is a projection in $\mathcal{M}$, then $p\mathcal{M}p$ will have trace $\tau(p)^{-1}\tau|_{p\mathcal{M}p}$.

Also, if A is an algebra with specified trace, $\overset{o}{A}$ will denote the ensemble of elements of A whose trace is zero.

First we examine $L(\mathbf{Z}_2)*L(\mathbf{Z}_2)$, (where $\mathbf{Z}_2$ is the two element group). The fact that $\mathcal{M} = L(\mathbf{Z}_2*\mathbf{Z}_2) \cong L(\mathbf{Z})\otimes M_2$ is well known, but we will need the following picture of $\mathcal{M}$.

Proposition 3.2. *Consider $\mathcal{M} = L(\mathbf{Z}_2) * L(\mathbf{Z}_2)$ with trace τ, and let p and q be projections of trace $\frac{1}{2}$ generating the first and respectively the second copy of $L(\mathbf{Z}_2)$. Then*

$$\mathcal{M} \cong L^\infty([0, \tfrac{\pi}{2}], \nu) \otimes M_2, \tag{10}$$

where ν is a probability measure on $[0, \frac{\pi}{2}]$ without atoms and τ is given by integration with respect to ν tensored with the normalized trace on $M_2 = M_2(\mathbf{C})$. Moreover, in the setup of (10), we have that

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } q = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \tag{11}$$

where $\theta \in [0, \frac{\pi}{2}]$.

Proof. It is well known that the universal unital C*–algebra generated by two projections p and q is $A = \{f : [0, \frac{\pi}{2}] \to M_2(\mathbf{C}) \mid f(0) \text{ and } f(\frac{\pi}{2}) \text{ diagonal}\}$, with p and q as in (11). $\mathcal{M}$ thus has a dense subalgebra equal to a quotient of A, and τ gives a trace on A. One can easily see that a trace on A must be of the following form. Let $f(t)_1$ and $f(t)_2$ be the diagonal values of $f(t)$ for $t = 0$ or $\frac{\pi}{2}$. Then

$$\tau(f) = a_1 f(0)_1 + a_2 f(0)_2 + \int_0^{\frac{\pi}{2}} \tau_2(f(t))\,d\nu(t) + b_1 f(\tfrac{\pi}{2})_1 + b_2 f(\tfrac{\pi}{2})_2,$$

where τ_2 is the normalized trace on $M_2(\mathbf{C})$, ν is a positive measure on $[0, \frac{\pi}{2}]$, $a_1, a_2, b_1, b_2 \geq 0$ and $|\nu|+a_1+a_2+b_1+b_2 = 1$. By Example 2.8 of [9], the distribution of pqp in $p\mathcal{M}p$ has no atoms, which implies that $|\nu| = 1$ and ν has no atoms. □

Remark 3.3. In the right hand side of (10), let

$$x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathrm{pol}((1-p)qp),$$

where "pol" means "polar part of." Then x is a partial isometry from p to $1-p$ and $\mathcal{M}$ is generated by pqp together with x. Let $y = \mathrm{pol}((1-q)pq)$. Then y is a partial isometry from q to $1-q$. Let

$$w = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Then w is unitary and $wpw^* = q$, $wxw^* = y$.

Definition 3.4. Let $(S_\iota)_{\iota\in I}$ be subsets of a unital algebra $A \ni 1$. A nontrivial *traveling product* in $(S_\iota)_{\iota\in I}$ is a product $a_1a_2\cdots a_n$ such that $a_j \in S_{\iota_j}$ $(1 \leq j \leq n)$ and $\iota_1 \neq \iota_2 \neq \iota_3 \neq \cdots \neq \iota_n$. The *trivial traveling product* is the identity element 1. $\Lambda((S_\iota)_{\iota\in I})$ denotes the set of all traveling products in $(S_\iota)_{\iota\in I}$, including the trivial one. If $|I| = 2$, we will often call traveling products *alternating products*.

Theorem 3.5. *Let A and B be finite von Neumann algebras (with specified faithful traces—see Remark 3.1). Then*

$$\begin{array}{llcl} (i) & (A\otimes L(\mathbf{Z}_2))*(B\otimes L(\mathbf{Z}_2)) & \cong & (A*A*B*B*L(\mathbf{Z}))\otimes M_2, \\ (ii) & (A\otimes M_2)*(B\otimes L(\mathbf{Z}_2)) & \cong & (A*B*B*L(\mathbf{F}_2))\otimes M_2, \\ (iii) & (A\otimes M_2)*(B\otimes M_2) & \cong & (A*B*L(\mathbf{F}_3))\otimes M_2. \end{array}$$

Proof. Let $\mathcal{M}$ be the von Neumann algebra on the left hand side of (i) with trace τ. It will be notationally convenient to identify A with $A\otimes 1\subseteq\mathcal{M}$ and B with $B\otimes 1\subseteq\mathcal{M}$. Let p and q be projections of trace $\frac{1}{2}$ contained in the copy of $1\otimes L(\mathbf{Z}_2)$ that commute with A and respectively B. Let $\mathcal{N}_0=\{p,q\}''\cong L(\mathbf{Z}_2)*L(\mathbf{Z}_2)$, and let $x,y,w\in\mathcal{N}_0$ be as in Remark 3.3. Then

$$p\mathcal{M}p=\big(\{pqp\}\cup pA\cup x^*Ax\cup w^*qBw\cup w^*y^*Byw\big)''.$$

We claim moreover that $\{\{pqp\},pA,x^*Ax,w^*qBw,w^*y^*Byw\}$ is a free family in $p\mathcal{M}p$, which then clearly implies (i).

Let us first show that $\{\{pqp\},pA,x^*Ax\}$ is free in $p\mathcal{M}p$. Let $g_k=(pqp)^k-2\tau((pqp)^k)p$, $(k\geq 1)$. To show freeness means to show that a nontrivial traveling product in $\{g_k\mid k\geq 1\}$, $p\overset{o}{A}$ and $x^*\overset{o}{A}x$ has trace zero. Regrouping gives a traveling product in $\Omega_0=\{x,x^*\}\cup\{g_k,xg_k,g_kx^*,xg_kx^*\mid k\geq 1\}$ and $\overset{o}{A}$. Let $a=p-\frac{1}{2}$, $b=q-\frac{1}{2}$. Then $\mathcal{N}_0=\{a,b\}''$, and $\operatorname{span}\Lambda(\{a\},\{b\})$ is a dense $*$–subalgebra of $\mathcal{N}_0$. Note that $\Omega_o\subset\mathcal{N}_0$, so that by the Kaplansky Density Theorem, any $z\in\Omega_0$ is the s.o.–limit of a bounded sequence in $\operatorname{span}\Lambda(\{a\},\{b\})$. Note also that since a and b are free and each has trace zero, the trace of an element of $\operatorname{span}\Lambda(\{a\},\{b\})$ is equal to the coefficient of 1. Since $\tau(z)=0$, we may chose that approximating sequence in $\operatorname{span}\Lambda(\{a\},\{b\})$ so that each coefficient of 1 equals zero. Moreover, since also $\tau(pz)=0$, we may also insist that each coefficient of a be zero, *i.e.* we have a bounded approximating sequence for z of elements of $\operatorname{span}(\Lambda(\{a\},\{b\})\backslash\{1,a\})$. We must now only show that a nontrivial alternating product in $\Lambda(\{a\},\{b\})\backslash\{1,a\})$ and $\overset{o}{A}$ has trace zero. Regrouping gives a nontrivial alternating product in $\{a\}\cup\overset{o}{A}\cup a\overset{o}{A}$ and $\{b\}$, which by freeness has trace zero.

Let $\mathcal{N}_1=(A\cup\mathcal{N}_0)''$, and let us show that $\{qw\mathcal{N}_1w^*,qB,y^*By\}$ is free in $q\mathcal{M}q$, which will complete the proof of (i). We show that a nontrivial traveling product in $w\mathcal{N}_1w^*$, $q\overset{o}{B}$ and $y^*\overset{o}{B}y$ has trace zero. Regrouping gives a traveling product in $\Omega_1=\{y,y^*\}\cup qw\overset{o}{\mathcal{N}}_1w^*\cup yw\overset{o}{\mathcal{N}}_1w^*\cup w\overset{o}{\mathcal{N}}_1w^*y^*\cup yw\overset{o}{\mathcal{N}}_1w^*y^*$ and $\overset{o}{B}$. Now $\Omega_1\subset\mathcal{N}_1$, $\operatorname{span}\Lambda(\{a\}\cup\overset{o}{A}\cup a\overset{o}{A},\{b\})$ is a dense $*$–subalgebra of $\mathcal{N}_1$ and $\tau(z)=\tau(qz)=0\ \forall\ z\in\Omega_1$, so that as above, each $z\in\Omega_1$ is the s.o.–limit of a bounded sequence in $\operatorname{span}(\Lambda(\{a\}\cup\overset{o}{A}\cup a\overset{o}{A},\{b\})\backslash\{1,b\})$. So it suffices to show that a nontrivial alternating product in $\operatorname{span}(\Lambda(\{a\}\cup\overset{o}{A}\cup a\overset{o}{A},\{b\})\backslash\{1,b\})$ and $\overset{o}{B}$ has trace zero. Regrouping gives a nontrivial alternating product in $\{a\}\cup\overset{o}{A}\cup a\overset{o}{A}$ and $\{b\}\cup\overset{o}{B}\cup b\overset{o}{B}$, which by freeness has trace zero.

Now we prove (ii). Let $\mathcal{M}$ be the von Neumann algebra on the left hand side of (ii), and let τ be its trace. We will identify A with $A\otimes 1$ and B with $B\otimes 1$ as in the proof of (i). Let p be a projection in $1\otimes M_2$ (commuting with A) of trace $\frac{1}{2}$ and q a projection in $1\otimes L(\mathbf{Z}_2)$ (commuting with B) of trace $\frac{1}{2}$. Let $\mathcal{N}_0=\{p,q\}''$ and let $x,y,w,a,b\in\mathcal{N}_0$ be as in the proof of (i). Let $u\in 1\otimes M_2$ be a partial isometry from p to $1-p$. Then

$$p\mathcal{M}p=\big(\{pqp,x^*u\}\cup pA\cup w^*qBw\cup w^*y^*Byw\big)'',$$

and we shall show that x^*u is a Haar unitary (*i.e.* a unitary such that $(x^*u)^n$ has trace zero $\forall\ n\in\mathbf{Z}\backslash\{0\}$) and that $\big\{\{pqp\},\{x^*u\},pA,w^*qBw,w^*y^*Byw\big\}$ is $*$–free in $p\mathcal{M}p$. This will in turn prove (ii). For $n>0$, $r=(x^*u)^n$ is a nontrivial alternating product in $\{x^*\}$ and $\{u\}$, and x^* is the s.o.–limit of a bounded sequence in $\operatorname{span}(\Lambda(\{a\},\{b\})\backslash\{1,a\})$, so to show $\tau(r)=0$ it suffices to show that a nontrivial alternating product in $\operatorname{span}(\Lambda(\{a\},\{b\})\backslash\{1,a\})$ and $\{u\}$ has trace zero. Regrouping gives a nontrivial alternating product in $\{a,u\}$ and $\{b\}$, which by freeness has trace zero. Hence we have shown that x^*u is a Haar unitary in $p\mathcal{M}p$.

Now we show that x^*u and pqp are $*$–free in $p\mathcal{M}p$. Let g_k $(k\geq 1)$ be as in the proof of (i). It suffices to show that a nontrivial alternating product in $\{(x^*u)^n\mid n\in\mathbf{Z}\backslash\{0\}\}$ and $\{g_k\mid k\geq 1\}$ has trace zero. Regrouping gives

an alternating product in Ω_0 and $\{u,u^*\}$, where Ω_0 is as in the proof of (i), which, proceeding as we did above, we see has trace zero. Similarly, we can show that letting $\widetilde{\mathcal{N}}_0=\{pqp,x^*u\}''$, $\{\widetilde{\mathcal{N}},pA\}$ is free in $p\mathcal{M}p$, and that letting $\widetilde{\mathcal{N}}_1=(\widetilde{\mathcal{N}}_0\cup A)''$, $\{w^*\widetilde{\mathcal{N}}_1w,qB,y^*By\}$ is free in $q\mathcal{M}q$, thus proving (ii).

To prove (iii), let p and u in $1\otimes M_2$ commuting with A be as above, let $q\in 1\otimes M_2$ commuting with B be a projection of trace $\frac{1}{2}$ and $v\in 1\otimes M_2$ commuting with B a partial isometry from q to $1-q$. Let $x,y,w\in\mathcal{N}_0=\{p,q\}''$ be as above. Then we similarly show that x^*u and y^*v are Haar unitaries and that $\big\{\{pqp\},\{x^*u\},pA,\{w^*y^*vw\},w^*qBw,\big\}$ is $*$–free in $p\mathcal{M}p$, (and notice that these taken together generate $p\mathcal{M}p$), which proves (iii). □

Corollary 3.6. *Let R and $\widetilde{R}$ be copies of the hyperfinite II_1–factor. Then*

$$R*\widetilde{R}\cong R*L(\mathbf{Z}),$$

with an isomorphism which when restricted is the identity map from R to R.

Proof. Write $R=(pRp)\otimes M_2$ and $\widetilde{R}=(\tilde{p}\widetilde{R}\tilde{p})\otimes M_2$, where p and $\tilde{p}$ are projections of trace $\frac{1}{2}$ in R and respectively $\widetilde{R}$. Then by (iii) and the proof of (iii),

$$p(R*\widetilde{R})p\cong(pRp)*(\tilde{p}\widetilde{R}\tilde{p})*L(\mathbf{F}_3),$$

and the isomorphism when restricted to $pRp\subset p(R*\widetilde{R})p$ is the identity map from pRp to pRp. Similarly, writing also $L(\mathbf{Z})\cong L(\mathbf{Z})\otimes L(\mathbf{Z}_2)$, we have from (ii) and the proof of (ii) that

$$p(R*L(\mathbf{Z}))p\cong(pRp)*L(\mathbf{F}_4),$$

and the isomorphism, when restricted to $pRp\subset p(R*L(\mathbf{Z}))p$, is the identity map from pRp to pRp. Considering the isomorphism (3), we get an isomorphism from $p(R*\widetilde{R})p$ to $p(R*L(\mathbf{Z}))p$ which when restricted is the identity map on pRp. Now tensor with M_2. □

§4. The addition formula for free products.

Theorem 4.1. $L(\mathbf{F}_r)*L(\mathbf{F}_{r'})=L(\mathbf{F}_{r+r'})$ *for* $1<r,r'\le\infty$.

Proof. (Please see the comments at the end of the introduction.) In a W*–probability space $(\mathcal{M},\tau)$ where τ is a trace, let R and $\widetilde{R}$ be copies of the hyperfinite II$_1$–factor and let $\nu=\{X^t\mid t\in T\}$ be a semicircular family such that $\{R,\widetilde{R},\nu\}$ is free. Let

$$L(\mathbf{F}_r)=\mathcal{A}=\big(R\cup\{p_sX^sp_s\mid s\in S\}\big)'',$$
$$L(\mathbf{F}_{r'})=\mathcal{B}=\big(\widetilde{R}\cup\{q_sX^sq_s\mid s\in S'\}\big)'',$$

where S and S' are disjoint subsets of T, $p_s\in R$, $q_s\in\widetilde{R}$ are projections and where $1+\sum_{s\in S}\tau(p_s)^2=r$, $1+\sum_{s\in S'}\tau(q_s)^2=r'$. Then $\mathcal{A}$ and $\mathcal{B}$ are free in $(\mathcal{M},\tau)$, so

$$L(\mathbf{F}_r)*L(\mathbf{F}_{r'})\cong\mathcal{N}=\big(R\cup\widetilde{R}\cup\{p_sX^sp_s\mid s\in S\}\cup\{q_sX^sq_s\mid s\in S'\}\big)''.$$

By Corollary 3.6, there exists a semicircular element $Y\in\mathcal{N}_0=(R\cup\widetilde{R})''$ such that R and $\{Y\}$ are free and together they generate $\mathcal{N}_0$. Moreover, for $s\in S'$ let $U_s\in\mathcal{N}_0$ be a unitary such that $U_sq_sU_s^*=f_s\in R$. Then

$$\mathcal{N}=\big(R\cup\{Y\}\cup\{p_sX^sp_s\mid s\in S\}\cup\{f_s(U_sX^sU_s^*)f_s\mid s\in S'\}\big)''.$$

To prove the theorem, it suffices to observe that $\{R,\{Y\},(\{X^s\})_{s\in S},(\{U_sX^sU_s^*\})_{s\in S'}\}$ is free in $\mathcal{M}$. □

Let us recall [4] that the fundamental group of a II$_1$–factor $\mathcal{M}$ is defined to be the set of positive real numbers γ such that $\mathcal{M}_\gamma\cong\mathcal{M}$. Murray and von Neumann [4] showed that the fundamental group of the hyperfinite II$_1$–factor is $\mathbf{R}_+$, and recently Rădulescu [5] has shown that the fundamental group of $L(\mathbf{F}_\infty)$ is also $\mathbf{R}_+$. A. Connes [1] has shown that the fundamental group of $L(G)$ where G is a group with property T of Kazhdan must be countable, but no other examples are known for fundamental groups of II$_1$–factors.

Equation (2) shows that the isomorphism question for (interpolated) free group factors is equivalent to the fundamental group question. Combined with the addition formula for free products, we now see that we must have one of two extremes.

Corollary 4.2. *We must have either*

(I) $L(\mathbf{F}_r) \cong L(\mathbf{F}_{r'})$ *for all* $1 < r, r' < \infty$ *and the fundamental group of* $L(\mathbf{F}_r)$ *is* $\mathbf{R}_+$ *for all* $1 < r < \infty$,

or (II) $L(\mathbf{F}_r) \not\cong L(\mathbf{F}_{r'})$ *for all* $1 < r < r' < \infty$ *and the fundamental group of* $L(\mathbf{F}_r)$ *is* $\{1\}$ *for all* $1 < r < \infty$.

Proof. Using formulas (1) and (2) we can show that if $L(\mathbf{F}_r) = L(\mathbf{F}_{r'})$ for some $r \neq r'$, then we have $L(\mathbf{F}_r) = L(\mathbf{F}_{r''})$ for r'' in some open interval, hence that the fundamental group of $L(\mathbf{F}_r)$ contains an open interval, thus is all of $\mathbf{R}_+$. □

Acknowledgements.

I would like to thank Dan Voiculescu, my advisor, for helpful discussions and for suggesting I look at free products such as $M_2(\mathbf{C}) * M_2(\mathbf{C})$.

References

1. A. Connes, *A factor of type II_1 with countable fundamental group*, J. Operator Theory **4** (1980), 151–153.
2. K. Dykema, *On certain free product factors via an extended matrix model*, J. Funct. Anal. (to appear).
3. ______, *Free products of hyperfinite von Neumann algebras and free dimension*, Duke Math. J. (to appear).
4. F.J. Murray and J. von Neumann, *Rings of operators. IV*, Ann. of Math. **44** (1943), 716–808.
5. F. Rădulescu, *The fundamental group of the von Neumann algebra of a free group with infinitely many generators is* $\mathbf{R}_+\backslash\{0\}$, J. Amer. Math. Soc. (to appear).
6. ______, *Stable isomorphism of the weak closure of free group convolution algebras*, preprint, I.H.E.S., December, 1991.
7. ______, *Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group*, preprint, I.H.E.S., December, 1991.
8. D. Voiculescu, *Symmetries of some reduced free product C^*–algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics, Volume 1132, Springer–Verlag, 1985, pp. 556–588.
9. ______, *Multiplication of certain non–commuting random variables*, J. Operator Theory **18** (1987), 223-235.
10. ______, *Noncommutative random variables and spectral problems in free product C^*–algebras*, Rocky Mt. J. Math. **20** (1990), 263-283.
11. ______, *Circular and semicircular systems and free product factors*, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Progress in Mathematics, Volume 92, Birkhäuser, Boston, 1990.
12. ______, *Limit laws for Random matrices and free products*, Invent. Math. **104** (1991), 201-220.
13. ______, *Free non-commutative random variables, random matrices and the II_1–factors of free groups*, Quantum Probability and Related Topics VI, L. Accardi, ed., World Scientific, Singapore, 1991, pp. 473–487.